Noncommutative geometrical structures of entangled quantum states

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Abstract

We study the noncommutative geometrical structures of quantum entangled states. We show that the space of a pure entangled state is a noncommutative space. In particular we show that by rewritten the conifold or the Segre variety we can get a q-deformed relation in noncommutative geometry. We generalized our construction into a multi-qubit state. We also in detail discuss the noncommutative geometrical structure of a three-qubit state.

1 Introduction

Quantum entangled states are the main resources in the field of quantum information science. These states also have very rich geometrical and topological structures. Geometrically the space of a pure quantum state is a complex projective space, that is $\mathcal{PH} = \mathcal{H}/\sim$, where $\mathcal{H}$ is the Hilbert space and $\sim$ is a equivalence relation. For example, if we let $\mathcal{H} = \mathbb{C}^{n+1}$, then $\mathcal{PH} = \mathbb{CP}^n$. Recently, we also have established relation between multi-projective variety (space) and pure quantum multipartite state. We have shown that the multi-projective Segre variety is the space of separable quantum composite systems and so it can distinguish between separable and entangled multipartite quantum systems [1]. Topologically, the space of two level state or a qubit can be described by Block sphere $S^2 \cong \mathbb{CP}^1$ which is obtained by Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2$. Generally, we have the following Hopf fibration $S^{2n+1} \longrightarrow \mathbb{CP}^n$. We also shown that there is relation between Hopf fibration and multi-qubit states [2]. For a pure multi-qubit state

$$
\begin{aligned}
|\Psi\rangle &= \sum_{x_m=0}^{1} \sum_{x_{m-1}=0}^{1} \cdots \sum_{x_1=0}^{1} \alpha_{x_m x_{m-1} \cdots x_1} |x_m x_{m-1} \cdots x_1\rangle, \qquad (1) \\
&= \sum_{x_m=0}^{1} \sum_{x_{m-1}=0}^{1} \cdots \sum_{x_1=0}^{1} \alpha_x |x\rangle,
\end{aligned}
$$

where $|x_m x_{m-1} \cdots x_1\rangle = |x_m\rangle \otimes |x_{m-1}\rangle \otimes \cdots \otimes |x_1\rangle$ are orthonormal basis in $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$ and $x = x_{m-1}2^{m-1} + x_{m-2}2^{m-2} + \cdots + x_0 2^0$, the set of state is defined by $\mathcal{SH}_{\mathcal{Q}} = \{|\Psi\rangle \in \mathcal{H}_{\mathcal{Q}} : \langle\Psi|\Psi\rangle = 1\}$. In this paper, we establish a relation between noncommutative geometry and quantum entangled

state. In particular, we show that by resolving the singularity of conifold we get a space which can be written in such form that is a q-deformed relation in noncommutative geometry. In section 2 we give a short introduction to multi-projective variety and in section 3 we review the construction of conifold and the quantum plane. In section we establish our first result, namely, the noncommutative structure two-qubit state. Finally, in section we generalize our result into multi-qubit state and we also discuss the three-qubit state as an illustrative example.

2 Multi-projective variety

In this section, we will review the construction of projective variety and in particular the multi-projective Segre variety. Here are some prerequisites on projective algebraic geometry [3, 4]. Let $\mathbb{C}$ be a complex algebraic field. Then, an affine n-space over $\mathbb{C}$ denoted $\mathbb{C}^n$ is the set of all n-tuples of elements of $\mathbb{C}$. An element $P \in \mathbb{C}^n$ is called a point of $\mathbb{C}^n$ and if $P = (a_1, a_2, \ldots, a_n)$ with $a_j \in \mathbb{C}$, then a_j is called the coordinates of P.

Let $\mathbb{C}[z] = \mathbb{C}[z_1, z_2, \ldots, z_n]$ denotes the polynomial algebra in n variables with complex coefficients. Then, given a set of q polynomials $\{g_1, g_2, \ldots, g_q\}$ with $g_i \in \mathbb{C}[z]$, we define a complex affine variety as

$$\mathcal{V}_{\mathbb{C}}(g_1, g_2, \ldots, g_q) = \{P \in \mathbb{C}^n : g_i(P) = 0 \ \forall\ 1 \leq i \leq q\}, \tag{2}$$

A complex projective space $\mathbb{CP}^n$ is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$, that is,

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - 0}{(u_1, \ldots, u_{n+1}) \sim (v_1, \ldots, v_{n+1})}, \ \lambda \in \mathbb{C} - 0, \ v_i = \lambda u_i \ \forall\ 0 \leq i \leq n+1. \tag{3}$$

Given a set of homogeneous polynomials $\{g_1, g_2, \ldots, g_q\}$ with $g_i \in \mathbb{C}[z]$, we define a complex projective variety as

$$\mathcal{V}(g_1, \ldots, g_q) = \{O \in \mathbb{CP}^n : g_i(O) = 0 \ \forall\ 1 \leq i \leq q\}, \tag{4}$$

where $O = [a_1, a_2, \ldots, a_{n+1}]$ denotes the equivalent class of point $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \in \mathbb{C}^{n+1}$. We can view the affine complex variety $\mathcal{V}_{\mathbb{C}}(g_1, g_2, \ldots, g_q) \subset \mathbb{C}^{n+1}$ as a complex cone over the complex projective variety $\mathcal{V}(g_1, g_2, \ldots, g_q)$.

We can map the product of spaces $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$ into a projective space by its Segre embedding as follows. Let (α_0^i, α_1^i) be points defined on the ith complex projective space $\mathbb{CP}^1$. Then the Segre map is given by

$$\begin{array}{rcc} \mathcal{S}_{2,\ldots,2} : \mathbb{CP}^1 \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^{2^m - 1} \\ ((\alpha_0^1, \alpha_1^1), \ldots, (\alpha_0^m, \alpha_1^m)) & \longmapsto & (\alpha_{i_m}^m \alpha_{i_{m-1}}^{m-1} \cdots \alpha_{i_1}^1). \end{array} \tag{5}$$

Now, let $\alpha_{i_m i_{m-1} \cdots i_1}$,$0 \leq i_s \leq 1$ be a homogeneous coordinate-function on $\mathbb{CP}^{2^m-1}$. Moreover, let us consider a multi-qubit quantum system and let $\mathcal{A} = \left(\alpha_{i_m i_{m-1} \ldots i_1}\right)_{0 \leq i_s \leq 1}$, for all $j = 1, 2, \ldots, m$. $\mathcal{A}$ can be realized as the following set $\{(i_1, i_2, \ldots, i_m) : 1 \leq i_s \leq 2, \forall\, s\}$, in which each point $(i_m, i_{m-1}, \ldots, i_1)$ is assigned the value $\alpha_{i_m i_{m-1} \ldots i_1}$. This realization of $\mathcal{A}$ is called an m-dimensional box-shape matrix of size $2 \times 2 \times \cdots \times 2$, where we associate to each such matrix

a sub-ring $\mathrm{S}_{\mathcal{A}} = \mathbb{C}[\mathcal{A}] \subset \mathrm{S}$, where S is a commutative ring over the complex number field. For each $s = 1, 2, \ldots, m$, a two-by-two minor about the j-th coordinate of $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{P}^{s}_{x_m y_m; x_{m-1} y_{m-1}; \ldots; x_1 y_1} &= \alpha_{x_m x_{m-1} \ldots x_1} \alpha_{y_m y_{m-1} \ldots y_1} \\ &- \alpha_{x_m x_{m-1} \ldots x_{s+1} y_s x_{s-1} \ldots x_1} \alpha_{y_m y_{m-1} \ldots y_{s+1} x_s y_{s-1} \ldots y_m} \in \mathrm{S}_{\mathcal{A}}. \end{aligned} \tag{6}$$

Then the ideal $\mathcal{I}^{m}_{\mathcal{A}}$ of $\mathrm{S}_{\mathcal{A}}$ is generated by $\mathcal{P}^{s}_{x_m y_m; x_{m-1} y_{m-1}; \ldots; x_1 y_1}$ and describes the separable states in $\mathbb{CP}^{2^m-1}$. The image of the Segre embedding $\mathrm{Im}(\mathcal{S}_{2,2,\ldots,2})$, which again is an intersection of families of quadric hypersurfaces in $\mathbb{CP}^{2^m-1}$, is called Segre variety and it is given by

$$\mathrm{Im}(\mathcal{S}_{2,2,\ldots,2}) = \bigcap_{\forall s} \mathcal{V}\left(\mathcal{P}^{s}_{x_m y_m; x_{m-1} y_{m-1}; \ldots; x_1 y_1}\right). \tag{7}$$

In the following section we establish relations between deformed Segre variety and q-deformed noncommutative geometry.

3 Conifold and quantum plane

In this section we will give a short review of conifold and quantum plane. An example of real (complex) affine variety is conifold which is defined by

$$\mathcal{V}_{\mathbb{C}}(z) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \sum_{i=1}^{4} z_i^2 = 0\}. \tag{8}$$

and conifold as a real affine variety is define by

$$\mathcal{V}_{\mathbb{R}}(f_1, f_2) = \{(u_1, \ldots, u_4, v_1, \ldots, v_4) \in \mathbb{R}^8 : \sum_{i=1}^{4} u_i^2 = \sum_{j=1}^{4} v_j^2, \sum_{i=1}^{4} u_i v_i = 0\}. \tag{9}$$

where $f_1 = \sum_{i=1}^{4}(u_i^2 - v_i^2)$ and $f_2 = \sum_{i=1}^{4} u_i v_i$. This can be seen by defining $z = u + iv$ and identifying imaginary and real part of equation $\sum_{i=1}^{4} z_i^2 = 0$. As a real space, the conifold is cone in $\mathbb{R}^8$ with top the origin and base space the compact manifold $\mathbb{S}^2 \times \mathbb{S}^3$. One can reformulate this relation in term of a theorem. The conifold $\mathcal{V}_{\mathbb{C}}(\sum_{i=1}^{4} z_i^2)$ is the complex cone over the Segre variety $\mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^3$. To see this let us make a complex linear change of coordinate

$$\begin{pmatrix} \alpha'_{00} & \alpha'_{01} \\ \alpha'_{10} & \alpha'_{11} \end{pmatrix} \longrightarrow \begin{pmatrix} z_1 + iz_2 & -z_4 + iz_3 \\ z_4 + iz_3 & z_1 - iz_2 \end{pmatrix}. \tag{10}$$

Thus after this linear coordinate transformation we have

$$\mathcal{V}_{\mathbb{C}}(\alpha'_{00}\alpha'_{11} - \alpha'_{01}\alpha'_{10}) = \mathcal{V}_{\mathbb{C}}(\sum_{i=1}^{4} z_i^2) \subset \mathbb{C}^4. \tag{11}$$

Thus we can think of conifold as a complex cone over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Moreover, we can remove the singularity of complex conifold $T^*\mathbb{S}^3$ with a global complex deformation parameter Ω. In this case we have a hypersurface $H = H(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ which is embedded in $\mathbb{C}^4$ by

$$\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = \alpha_0\alpha_3 - \alpha_1\alpha_2 = \Omega. \tag{12}$$

We will return to this equation in the following sections when we discuss non-commutative geometrical structures of two-qubits.

Next we will give a short introduction to quantum plane [5]. Let $\mathbb{C}$ be a complex number field and q be a invertible element of $\mathbb{C}$. Moreover, let I_q be the two side ideal of the free algebra $\mathbb{C}\{u, v\}$ which is generated by $vu - quv$. Then the quantum plane is defined to be the quotient-algebra

$$\mathbb{C}_q[u, v] = \mathbb{C}\{u, v\}/I_q. \tag{13}$$

The quantum plane is non-commutative if $q \neq 1$. The ideal I_q is generated by homogeneous degree two element. For any pair (i, j), we have $v^j u^i - qu^i v^j = 0$ and Given any $\mathbb{C}$-algebra R, there is a bijection

$$\mathrm{Hom}(\mathbb{C}_q[u, v], R) \cong \{(U, V) \in R \times R : VU - qUV = 0\}. \tag{14}$$

The pair (U, V) satisfying above relation are called a R-point of the quantum plane. There is direct relation between our construction in the following section and quantum plane. Our short review of quantum plane could be important in future investigation of quantum geometry and quantum entangled states.

4 Noncommutative geometrical structure of two-qubits

In this section we investigate a pure two-qubit state based on noncommutative geometry. A pure two-qubit state is given by

$$|\Psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle. \tag{15}$$

Now, based on the Segre variety construction, the separable state of such two-qubit state is given by

$$\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = 0 \tag{16}$$

Thus, for entangled state we have $\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} \neq 0$. As we have discussed this condition is also related to deformation of conifold. Now, let

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}. \tag{17}$$

Then, for this deformed variety we have

$$\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = u_1 v_2 - u_2 v_1 = \Omega \tag{18}$$

which in this form is a q-deformed relation in noncommutative geometry [6]. Now, let $\mu_i = (u_1, u_2)$ and $\nu_i = (v_1, v_2)$. We can also write this equations as $\varepsilon^{ij}\mu_i\nu_j = \Omega$, $\varepsilon^{ij}\mu_i\mu_j = 0$, and $\varepsilon^{ij}\nu_i\nu_j = 0$, where ε^{ij} is an antisymmetric tensor. Note that the relation $\varepsilon^{ij}\mu_i\nu_j = \Omega$ express $SL(2, \mathbb{C})$ invariance of conifold hypersurface H in complex space $\mathbb{C}^4$. We also can write these equations as

$$\mu_{[i}\nu_{j]} = \Phi_{ij}, \quad \mu_{[i}\nu_{j]} = 0\ , \quad \text{and} \quad \nu_{[i}\nu_{j]} = 0, \tag{19}$$

where $\Phi_{ij} = \varepsilon_{ij}\Omega/2$. Next, we set $\mu_i = \Lambda_{1i}$ and $\nu_i = \Lambda_{2i}$, then we get

$$\Lambda_{ki}\Lambda_{lj} - \Lambda_{kj}\Lambda_{li} = \varepsilon_{kl}\Phi_{ij} = \Lambda_{ki}\Lambda_{lj} - \mathfrak{R}^{mn}_{kl}\Lambda_{mj}\Lambda_{ni}, \tag{20}$$

where $\mathfrak{R}^{mn}_{kl} = \varepsilon^m_k \varepsilon^n_l$. This is a noncommutative space of a pure two-qubit entangled state. Moreover, this rewriting of deformed variety for two-qubit could alow us to borrow techniques and tools from theory of q-deformed noncommutative geometry to investigate the structures of entangled states.

5 Multi-qubit states

One would now ask if it possible to establish relation between multipartite quantum states and noncommutative geometry. The answer seems to be positive, since based on the multi-projective Segre variety, the complectly separable set of pure state is give by quadratic polynomial defined by equation (6). But, when discussing a multi-qubit state it is better to consider the Segre ideals,

$$\begin{array}{ccc}
\left(\begin{array}{cccc} \alpha_{00\cdots00} & \alpha_{00\cdots01} & \cdots & \alpha_{01\cdots11} \\ \alpha_{10\cdots00} & \alpha_{10\cdots01} & \cdots & \alpha_{11\cdots11} \end{array}\right) & = & \left(\begin{array}{cccc} u^{p_1}_1 & u^{p_1}_2 & \cdots & u^{p_1}_{2^{m-1}} \\ v^{p_1}_1 & v^{p_1}_2 & \cdots & v^{p_1}_{2^{m-1}} \end{array}\right), \\
\left(\begin{array}{cccc} \alpha_{00\cdots00} & \alpha_{00\cdots01} & \cdots & \alpha_{101\cdots1} \\ \alpha_{010\cdots0} & \alpha_{01\cdots01} & \cdots & \alpha_{11\cdots11} \end{array}\right) & = & \left(\begin{array}{cccc} u^{p_2}_1 & u^{p_2}_2 & u^{p_2}_3 & u^{p_2}_{2^{m-1}} \\ v^{p_2}_1 & v^{p_2}_2 & v^{p_2}_3 & v^{p_2}_{2^{m-1}} \end{array}\right), \\
 & \vdots & \\
\left(\begin{array}{cccc} \alpha_{00\cdots00} & \alpha_{0\cdots010} & \cdots & \alpha_{11\cdots10} \\ \alpha_{00\cdots01} & \alpha_{0\cdots011} & \cdots & \alpha_{11\cdots11} \end{array}\right) & = & \left(\begin{array}{cccc} u^{p_m}_1 & u^{p_m}_2 & \cdots & u^{p_m}_{2^{m-1}} \\ v^{p_m}_1 & v^{p_m}_2 & \cdots & v^{p_m}_{2^{m-1}} \end{array}\right).
\end{array} \tag{21}$$

Now we define

$$\mathrm{Minors}^{p_s}_{2\times 2}\left(\begin{array}{cccc} u^{p_s}_1 & u^{p_s}_2 & \cdots & u^{p_s}_{2^{m-1}} \\ v^{p_s}_1 & v^{p_s}_2 & \cdots & v^{p_s}_{2^{m-1}} \end{array}\right) = \Omega^{p_s}, \quad s = 1, 2, \ldots, m \tag{22}$$

where $p_s = \frac{2^{m-1}(2^{m-1}-1)}{2}$, is the number of quadratic polynomial defining the Segre variety of the multi-qubits and $\mathrm{Minors}^{p_s}_{2\times2}$ is the 2×2 minors of the above $2 \times 2^{m-1}$ matrices. Then, for example a multi-qubit deformed variety is given by

$$\alpha_{00\cdots0}\alpha_{10\cdots01} - \alpha_{0\cdots01}\alpha_{10\cdots00} = u^{p_1}_1 v^{p_1}_2 - u^{p_1}_2 v^{p_1}_1 = \Omega^{p_1}, \quad \text{for} \quad p_1 = 1 \tag{23}$$

which in this form is a q-deformed relation in noncommutative geometry, where p_s, for $s = 1, 2, \ldots, m$ is the number of quadratic polynomial defining the Segre variety of the multi-qubits. Now, let $\mu^{p_s}_{i_s} = (u^{p_s}_1, u^{p_s}_2, \ldots, u^{p_s}_{2^{m-1}})$ and $\nu^{p_s}_{i_s} = (v^{p_s}_1, v^{p_s}_2, \ldots, v^{p_s}_{2^{m-1}})$. Then, we can also write this equations as $\varepsilon^{i_s j_s}\mu^{p_s}_{i_s}\nu^{p_s}_{j_s} = \Omega^{p_s}$, $\varepsilon^{i_s j_s}\mu^{p_s}_{i_s}\mu^{p_s}_{j_s} = 0$, and $\varepsilon^{i_s j_s}\nu^{p_s}_{i_s}\nu^{p_s}_{j_s} = 0$ or as

$$\mu^{p_s}_{[i_s}\nu^{p_s}_{j_s]} = \Phi^{p_s}_{i_s j_s}, \quad \mu^{p_s}_{[i_s}\nu^{p_s}_{j_s]} = 0\ , \quad \text{and} \quad \nu^{p_s}_{[i_s}\nu^{p_s}_{j_s]} = 0, \tag{24}$$

where $\Phi^{p_s}_{i_s j_s} = \varepsilon_{i_s j_s}\Omega^{p_s}/2$. Next, following the same procedure, we let $\mu^{p_s}_{i_s} = \Lambda^{p_s}_{1 i_s}$ and $\nu^{p_s}_{i_s} = \Lambda^{p_s}_{2 i_s}$. Then we get

$$\Lambda^{p_s}_{k_s i_s}\Lambda^{p_s}_{l_s j_s} - \Lambda^{p_s}_{k_s j_s}\Lambda^{p_s}_{l_s i_s} = \varepsilon^{p_s}_{k_s l_s}\Phi^{p_s}_{i_s j_s} = \Lambda^{p_s}_{k_s i_s}\Lambda^{p_s}_{l_s j_s} - \mathfrak{R}^{m_s n_s}_{k_s l_s},\Lambda^{p_s}_{m_s j_s}\Lambda^{p_s}_{n_s i_s}, \tag{25}$$

where $\mathfrak{R}^{m_s n_s}_{k_s l_s} = \varepsilon^{m_s}_{k_s}\varepsilon^{n_s}_{l_s}$. To illustrate our construction we in detail discuss a three-qubit state which is given by $|\Psi\rangle = \sum^{1}_{x_3,x_2,x_1=0}\alpha_{x_3x_2x_1}|x_3x_2x_1\rangle$. Now, based on the Segre variety construction, the separable state of such three-qubit state is given by

$$\mathrm{Im}(\mathcal{S}_{2,2,2}) \quad = \quad \bigcap_{\forall s}\mathcal{V}\left(\mathcal{P}^{s}_{x_3y_3;x_2y_2;x_1y_1}\right). \tag{26}$$

Thus, for entangled state we have the following condition

$$\alpha_{x_3x_2x_1}\alpha_{y_3y_2y_1} - \alpha_{x_3y_sx_1}\alpha_{y_3x_sy_3} \neq 0. \tag{27}$$

As we have discussed this condition is also related to deformation of conifold. Since we can apply the same procedure to establish relation between quantum entangled states and noncommutative geometry as we have done for two-qubits. In this case we need to consider all quadratic polynomials $\alpha_{x_3x_2x_1}\alpha_{y_3y_2y_1} - \alpha_{x_3y_sx_1}\alpha_{y_3x_sy_3}$. We can also consider the Segre ideals for three-qubits,

$$\begin{pmatrix} \alpha_{000} & \alpha_{001} & \alpha_{010} & \alpha_{011} \\ \alpha_{100} & \alpha_{101} & \alpha_{110} & \alpha_{111} \end{pmatrix} = \begin{pmatrix} u_1^{p_1} & u_2^{p_1} & u_3^{p_1} & u_4^{p_1} \\ v_1^{p_1} & v_2^{p_1} & v_3^{p_1} & v_4^{p_1} \end{pmatrix}, \tag{28}$$

$$\begin{pmatrix} \alpha_{000} & \alpha_{001} & \alpha_{100} & \alpha_{101} \\ \alpha_{010} & \alpha_{011} & \alpha_{110} & \alpha_{111} \end{pmatrix} = \begin{pmatrix} u_1^{p_2} & u_2^{p_2} & u_3^{p_2} & u_4^{p_2} \\ v_1^{p_2} & v_2^{p_2} & v_3^{p_2} & v_4^{p_2} \end{pmatrix}, \tag{29}$$

$$\begin{pmatrix} \alpha_{000} & \alpha_{100} & \alpha_{010} & \alpha_{110} \\ \alpha_{001} & \alpha_{101} & \alpha_{011} & \alpha_{111} \end{pmatrix} = \begin{pmatrix} u_1^{p_3} & u_2^{p_3} & u_3^{p_3} & u_4^{p_3} \\ v_1^{p_3} & v_2^{p_3} & v_3^{p_3} & v_4^{p_3} \end{pmatrix}. \tag{30}$$

Now the equation (22) for a three-qubit system takes the following form

$$\mathrm{Minors}_{2\times 2}^{p_s} \begin{pmatrix} u_1^{p_s} & u_2^{p_s} & u_3^{p_s} & u_4^{p_s} \\ v_1^{p_s} & v_2^{p_s} & v_3^{p_s} & v_4^{p_s} \end{pmatrix} = \Omega^{p_s}, \quad s = 1, 2, 3, \tag{31}$$

where $p_s = \frac{2^{3-1}(2^{3-1}-1)}{2} = 6$, is the number of quadratic polynomial defining the Segre variety of the three-qubits and $\mathrm{Minors}_{2\times 2}^{p_s}$ is the 2×2 minors of the above $2 \times 2^{3-1}$ matrices. In this form we have again a q-deformed relation in noncommutative geometry. For instance a deformed variety is given by

$$\alpha_{000}\alpha_{101} - \alpha_{001}\alpha_{100} = u_1^{p_1}v_2^{p_1} - u_2^{p_1}v_1^{p_1} = \Omega^{p_1}, \quad \text{for} \quad p_1 = 1. \tag{32}$$

Now, let $\mu_{i_s}^{p_s} = (u_1^{p_s}, u_2^{p_s}, u_3^{p_s}, u_4^{p_s})$ and $\nu_{i_s}^{p_s} = (v_1^{p_s}, v_2^{p_s}, v_3^{p_s}, v_4^{p_s})$. Then we have $\varepsilon^{ij}\mu_{i_s}^{p_s}\nu_{j_s}^{p_s} = \Omega^{p_s}$, $\varepsilon^{ij}\mu_{i_s}^{p_s}\mu_{j_s}^{p_s} = 0$, and $\varepsilon^{ij}\nu_{i_s}^{p_s}\nu_{j_s}^{p_s} = 0$. We can also write these equations as

$$\mu_{[i_s}^{p_s}\nu_{j]}^{p_s} = \Phi_{i_sj_s}^{p_s} = \varepsilon_{i_sj_s}\Omega^{p_s}/2, \quad \mu_{[i_s}^{p_s}\nu_{j_s]}^{p_s} = 0 \ , \quad \text{and} \quad \nu_{[i_s}^{p_s}\nu_{j]}^{p_s} = 0, \tag{33}$$

If, we set $\mu_{i_s}^{p_s} = \Lambda_{1i_s}^{p_s}$ and $\nu_{i_s}^{p_s} = \Lambda_{2i_s}^{p_s}$, then we get the following set of q-deformed relations

$$\begin{aligned}
\Lambda_{k_1i_1}^{p_1}\Lambda_{l_1j_1}^{p_1} - \Lambda_{k_1j_1}^{p_1}\Lambda_{l_1i_1}^{p_1} &= \varepsilon_{k_1l_1}^{p_1}\Phi_{i_1j_1}^{p_1} = \Lambda_{k_1i_1}^{p_1}\Lambda_{l_1j_1}^{p_1} - \mathfrak{R}_{k_1l_1}^{m_1n_1}\Lambda_{m_1j_1}^{p_1}\Lambda_{n_1i_1}^{p_1}, \\
\Lambda_{k_2i_2}^{p_2}\Lambda_{l_2j_2}^{p_2} - \Lambda_{k_2j_2}^{p_2}\Lambda_{l_2i_2}^{p_2} &= \varepsilon_{k_2l_2}^{p_2}\Phi_{i_2j_2}^{p_2} = \Lambda_{k_2i_2}^{p_2}\Lambda_{l_2j_2}^{p_2} - \mathfrak{R}_{k_2l_2}^{m_2n_2}\Lambda_{m_2j_2}^{p_2}\Lambda_{n_2i_2}^{p_2}, \\
\Lambda_{k_3i_3}^{p_3}\Lambda_{l_3j_3}^{p_3} - \Lambda_{k_3j_3}^{p_3}\Lambda_{l_3i_3}^{p_3} &= \varepsilon_{k_3l_3}^{p_3}\Phi_{i_3j_3}^{p_3} = \Lambda_{k_3i_3}^{p_3}\Lambda_{l_3j_3}^{p_3} - \mathfrak{R}_{k_3l_3}^{m_3n_3}\Lambda_{m_3j_3}^{p_3}\Lambda_{n_3i_3}^{p_3},
\end{aligned} \tag{34}$$

where $\mathfrak{R}_{k_1l_1}^{m_1n_1} = \varepsilon_{k_1}^{m_1}\varepsilon_{l_1}^{n_1}$, $\mathfrak{R}_{k_2l_2}^{m_2n_2} = \varepsilon_{k_2}^{m_2}\varepsilon_{l_2}^{n_2}$, and $\mathfrak{R}_{k_3l_3}^{m_3n_3} = \varepsilon_{k_3}^{m_3}\varepsilon_{l_3}^{n_3}$.

In this paper we have investigate the noncommutative structures of entangled quantum systems. First we have shown that the space of entangled two-qubits can be seen as deformed conifold. Then we wrote the coordinate of this variety in terms of noncommutative space. We have also discussed multipartite entangled systems in terms of noncommutative geometry. We belief that our construction not only important in foundation of quantum theory but it could give rise to new results and applications in the field of quantum information and quantum computing.

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References

[1] H. Heydari and G. Björk, J. Phys. A: Math. Gen. **38**, 3203-3211 (2005).

[2] H. Heydari, Quantum Information and Computation 6 (2006) 400-409.

[3] P. Griffiths and J. Harris, *Principle of algebraic geometry*, Wiley and Sons, New York, 1978.

[4] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer-Verlag, Berlin, 1976.

[5] C. Kassel, *Quantum groups*, Springer, Berlin-Heidelberg, 1995.

[6] C. Closset, lectures given at the Modave Summer School in Mathematical Physics 2008, e-print 0901.3695v2.